Master of Science (Mathematics)
Semester – II

Paper Code –

MEASURE AND INTEGRATION THEORY

MASTER OF SCIENCE (MATHEMATICS)
Measure and Integration Theory (Semester–II)
Paper code:

M. Marks = 100
Term End Examination = 80
Time = 3 hrs Assignment = 20

Course Outcomes

Students would be able to:

CO1 Describe the shortcomings of Riemann integral and benefits of Lebesgue integral.
CO2 Understand the fundamental concept of measure and Lebesgue measure.
CO3 Learn about the differentiation of monotonic function, indefinite integral, use of the fundamental theorem of calculus.

Section - I

Set functions, Intuitive idea of measure, Elementary properties of measure, Measurable sets and their fundamental properties. Lebesgue measure of a set of real numbers, Algebra of measurable sets, Borel set, Equivalent formulation of measurable sets in terms of open, Closed, F_σ and G_δ sets, Non measurable sets.

Section - II

Measurable functions and their equivalent formulations. Properties of measurable functions. Approximation of a measurable function by a sequence of simple functions, Measurable functions as nearly continuous functions, Egoroff theorem, Lusin theorem, Convergence in measure and F. Riesz theorem. Almost uniform convergence.

Section - III

Shortcomings of Riemann Integral, Lebesgue Integral of a bounded function over a set of finite measure and its properties. Lebesgue integral as a generalization of Riemann integral, Bounded convergence theorem, Lebesgue theorem regarding points of discontinuities of Riemann integrable functions, Integral of non-negative functions, Fatou Lemma, Monotone convergence theorem, General Lebesgue Integral, Lebesgue convergence theorem.

Section - IV

Vitali covering lemma, Differentiation of monotonic functions, Function of bounded variation and its representation as difference of monotonic functions, Differentiation of indefinite integral, Fundamental theorem of calculus, Absolutely continuous functions and their properties.

Note : The question paper of each course will consist of **five** Sections. Each of the sections **I to IV** will contain **two** questions and the students shall be asked to attempt **one** question from each. **Section-V** shall be **compulsory** and will contain **eight** short answer type questions without any internal choice covering the entire syllabus.

Books Recommended :

1. Walter Rudin, Principles of Mathematical Analysis (3rd edition) McGraw-Hill, Kogakusha, 1976, International Student Edition.
2. H.L. Royden, Real Analysis, Macmillan Pub. Co., Inc. 4th Edition, New York, 1993.
3. P. K. Jain and V. P. Gupta, Lebesgue Measure and Integration, New Age International (P) Limited Published, New Delhi, 1986.
4. G.De Barra, Measure Theory and Integration, Wiley Eastern Ltd., 1981.
5. R.R. Goldberg, Methods of Real Analysis, Oxford & IBH Pub. Co. Pvt. Ltd, 1976.
6. R. G. Bartle, The Elements of Real Analysis, Wiley International Edition, 2011.

Contents

SECTION – I

MEASURABLE SETS

Introduction

In measure theory, a branch of mathematics, the concept of **Lebesgue measure**, was given by French mathematician Henri Lebesgue in 1901. Sets that can be assigned a Lebesgue measure are called **Lebesgue-measurable**; the measure of the Lebesgue-measurable set *A* is here denoted by *m**(*A*).

Lebesgue Measure

In this section we shall define Lebesgue Measure, which is a generalization of the idea of length.

1.1 Definition. The length l(I) of an interval I with end points a and b is defined as the difference of the end points. In symbols, we write.

$$l\,(I) \;=\; b - a.$$

1.2 Definition. A function whose domain of definition is a class of sets is called a Set Function. For example, length is a set function. The domain being the collection of all intervals.

1.3 Definition. An extended real – valued set function μ defined on a class E of sets is called Additive if $A \in E, B \in E, A \cup B \in E$ and $A \cap B = \phi, imply$

$$\mu\,(A \cup B) \;=\; \mu\,(A) \;+\; \mu\,(B)$$

1.4 Definition. An extended real valued set function μ defined on a class E of sets is called finitely additive if for every finite disjoint classes $\{A_1, A_2, \ldots, A_n\}$of sets in E, whose union is also in E,

we have

$$\mu(U_{i=1}^{n} A_i\,) \;=\; \sum_{i=1}^{n} \mu(A_i)$$

1.5 Definition. An extended real–valued set function μ defined on a class E of sets is called countably additive it for every disjoint sequence $\{A_n\}$ of sets in E whose union is also in E, we have

$$\mu(U_{i=1}^{\infty} A_i\,) \;=\; \sum_{i=1}^{\infty} \mu(A_i)$$

1.6 Definition. Length of an open set is defined to be the sum of lengths of the open intervals of which it is composed of. Thus, if G is an open set, then

$$l(G) \;=\; \sum_{n} l(I_n)$$

where

$$G \;=\; U_n\, I_n\,, I_{n_1} \cap I_{n_2} \;=\; \phi \; if \; n_1 \neq n_2.$$

1.7 Definition. The Lebesgue Outer Measure or simply the outer measure m* of a set A is defined as

$$m^*(A) = \inf_{A \subseteq UI_n} \sum l(I_n).$$

where the infimum is taken over all finite or countable collections of intervals $\{I_n\}$ such that $A \subseteq UI_n$

Since the lengths are positive numbers, it follows from the definition of m^* that $m^*(A) \geq 0$.

1.8 Remark: (i) If $A \subseteq B$,then $m^*(A) \leq m^*(B)$ i.e. outer-measure has monotone property.

Proof: By definition of outer-measure, for each $\varepsilon > 0$, there exist a countable collection of open interval $\{I_n\}$ such that $B \subseteq U_n I_n$ and

$m^*(B) + \varepsilon > \sum_n l(I_n)$ ...1)

now $A \subseteq B$ *and* $B \subseteq U_n I_n$

$=> A \subseteq U_n I_n$

$m^*(A) \leq \sum_n l(I_n)$

$< m^*(B) + \varepsilon$ (*using* 1))

$\Rightarrow m^*(A) < m^*(B) + \varepsilon$

but $\varepsilon > 0$ is arbitrary, $m^*(A) \leq m^*(B)$ hence proved.

(ii) Outer-measure of a set is always non-negative.

1.9 Theorem. Outer measure is translation invariant.

Proof. Let $\epsilon > 0$ be given. Then by definition of outer measure, There exist a countable collection of intervals $\{I_n\}$ such that $A \subseteq \cup I_n$ *and*

$m^*(A) + \epsilon > \sum_n l(I_n)$.

Now, $A \subseteq \bigcup_n (I_n)$

$=> A + x \subseteq \bigcup_n (I_n + x)$,

$=> m^*(A + x) \leq \sum_n l(I_n + x) = \Sigma\, l(I_n)$ [length is translation invariant]

$\leq m^* A + \epsilon$

Since ϵ is arbitrary positive number, we have

(2) $m^*(A + x) \leq m^*(A)$ (1)

To prove reverse inequality, Let $\epsilon > 0$ be given. Then by definition of outer measure, There exist a countable collection of intervals $\{J_n\}$ such that

$A + x \subseteq \bigcup_n J_n$ *and*

$m^*(A + x) + \epsilon > \sum_n l(J_n)$.

Now, $A + x \subseteq \bigcup_n J_n$

⇨ $A \subseteq \cup_n (J_n - x)$

⇨ $m^* (A) \leq \sum_n l(J_n - x)$

⇨ $m^* (A) \leq \sum_n l(J_n) < m^* (A + x) + \in$

⇨ $m^* (A) \leq m^*(A + x)$ (2)

Then Combining (1) and (2), the required result follows.

i.e., $m^* (A) = m^*(A + x)$

1.10 Theorem. The outer measure of an interval is its length.

Proof. CASE (1) Let us suppose, first I is a closed and bounded interval, say I = [a, b]

To prove: m*(I) = ℓ [a, b] =b - a.

Now for each ε> 0, I = [a, b] ⊆ (a - ε, b+ ε) then

by definition of outer-measure

=> m*(I) ≤ ℓ (a - ε, b+ ε) ≤ (b+ ε- a+ ε)

=> m*(I) ≤ b-a + 2 ε

since ε is an arbitrary, m*(I) ≤ b-a = ℓ (I) (1)

Now to prove, m*(I) = b-a, then it is sufficient to prove m*(I) ≥ b-a. let $\{I_n\}$ be a countable collection of open intervals which covering I i.e.

I ⊆ $\cup_n I_n$

$\sum_n \ell(I_n) \geq$ b-a for all n ∈ N so it is sufficient to prove that

inf $\sum_n \ell(I_n) \geq$ b-a

since I = [a, b] is compact, then by Heine Boral theorem, we can select a finite number of open intervals from this $\{I_n\}$ such that their union contains I.

Let the intervals be J_1, J_2, ..., Jp such that $\cup_{i=1}^{p} J_i \supseteq$ [a, b].

Now it is sufficient to prove $\sum_{i=1}^{p} \ell(J_i) \geq$ b-a (2)

Now a ∈ I = [a, b], there exist open interval $J_1 = (a_1, b_1)$ from the above-mentioned finite no. of intervals such that $a_1 < a \leq b$ then $b_1 \in I$.

Again, there exist an open interval (a_2, b_2) from the finite collection J_1, J_2, ..., Jp such that $a_2 < b_1 < b_2$. Continuing this, we get a sequence of open intervals

(a_1, b_1), (a_2, b_2), ..., (a_p, b_p) from J_1, J_2, ..., Jp satisfying $a_i < b_{i-1} <$ bi ,i= 2,3,.....,p since the collection is finite so the process must stop with an interval satisfying $a_p < b_{p-1} < b_p$ and $a_p < b < b_p$

$\sum_n \ell(I_n) \geq \sum_{i=1}^{p} \ell(J_i) = \ell\,(a_1, b_1) + \ell\,(a_2, b_2) + \ldots.\ \ell\,(a_p, b_p)$

$= (b_1 - a_1) + (b_2 - a_2) + \ldots + (b_p - a_p)$

$= b_p + (b_{p-1} - a_p) + \ldots + b_1 - a_2 - a_1$

$> b_p - a_1$

$> b\text{-}a$

=> $\inf \sum_n \ell(I_n) \geq$ b-a

=> m*(I) ≥ b-a (4)

Hence result is proved in the case when I closed and bounded interval.

CASE (2) let I be bounded open interval with end points a and b, then for every real no. ε> 0 [a+ ε, b- ε] ⊂ I ⊂ [a, b]

=> m*[a+ ε, b- ε] ≤ m*(I) ≤ m* [a, b]

=> ℓ [a+ ε, b- ε] ≤ m*(I) ≤ ℓ [a, b] (by case 1)

=>b- ε -a- ε≤ m*(I) ≤b-a

since ε is arbitrary,

we get b-a ≤ m*(I) ≤ b-a

=> m*(I) =b-a.

CASE (3) if I is the unbounded interval, then for each real no. r> 0, we can find bounded closed interval J ⊂ I such that ℓ (J)>r

Now J ⊂ I => m* (J) ≤ m* (I)

=> ℓ (J) ≤ m* (I)

=> m* (I) > r since this hold for each real no. r,

we get m* (I) = ∞ = ℓ(I)

i.e. outer-measure is of an interval equal to its length.

1.11 Theorem. Let $\{A_n\}$ be a countable collection of sets of real numbers. Then $m^*(\cup A_n) \leq \Sigma\, m^* A_n$.

Proof. Proof. If one of the sets A_n has infinite outer measure, the inequality holds trivially. So suppose $m^*\{A_n\}$ is finite. Then, given $\in\, > 0$, there exists a countable collection $\{I_{n,i}\}$ of open intervals such that $A_n \subset U_i\, I_{n,i}$ and

$$\Sigma_i\ \ l(I_{n,i}) < m^*(A_n) + \frac{\varepsilon}{2^n}$$

by the definition of $m^*\{A_n\}$.

Now the collection $[I_{n,i}]_{n,i} = U_n\, [I_{n,i}]_i$ is countable, being the union of a countable number of countable collections, and covers $\bigcup_n A_n$. Thus

$$m^*\left(\bigcup_n A_n\right) \leq \Sigma_{n,i} l(I_{n,i})$$

$$= \Sigma_n \Sigma_i l(I_n, i)$$

$$< \left(m^*(A_n) + \frac{\in}{2^n} \right)$$

$$= \Sigma_n m^* A_n + \Sigma_n \frac{\in}{2^n}$$

$$= \Sigma_n m^* A_n + \epsilon \Sigma_n \frac{1}{2^n}$$

$$= \Sigma\, m^* A_n + \in$$

Since $\in$ is an arbitrary positive number, it follows that

$m^*(\cup_n A_n) \leq \Sigma\, m^*(A_n)$.

1.12 Theorem. Outer-measure of singleton set of reals is zero

Proof: Let $A = \{a\}$ Then, since $A = \{a\}, \{a\} \subseteq \left(a - \frac{1}{n}, a + \frac{1}{n}\right) \forall n \in N$

- ⇨ $m^*(a) \leq m * \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$
- ⇨ $m^*(a) \leq \frac{2}{n}$
- ⇨ $0 \leq m^*(a) \leq \frac{2}{n}$ for each n.

In limiting case $m^*(a) = 0$.

1.13 Theorem. Outer-measure of null set is zero.

Proof: Since $\phi \subseteq \left(-\frac{1}{n}, \frac{1}{n}\right) \forall n \in N$

- ⇨ $m^*(\phi) \leq m^*\left(-\frac{1}{n}, \frac{1}{n}\right)$
- ⇨ $m^*(\phi) \leq \frac{2}{n}$
- ⇨ $0 \leq m^*(\phi) \leq \frac{2}{n}$ for each n. In limiting case $m^*(\phi) = 0$.

1.14 Corollary. If A is countable, $m^* A = 0$

Proof. We know that a countable set is the union of a countable family of singleton. Therefore $A = \cup [x_n]$, which yields

$m^* A = m^* [\cup (x_n)] \leq \Sigma\, m^* [x_n]$ (by the above theorem)

But as already pointed out outer measure of a singleton is zero. Therefore it follows that

$$m^* A \leq 0$$

Since outer measure is always a non – negative real number, $m^* A = 0$.

1.15 Remark: The Sets $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}$ has outer-measure zero.

1.16 Remark: Prove that [0, 1] is uncountable.

Proof: Assume on the contrary that the set [0, 1] is countable, then as we know outer-measure of countable set is zero, then $m^*[0,1] = 0, i.e, l[0,1] = 0$

$i.e., 1 = 0$, a contradiction. therefore [0, 1] is uncountable.

1.17 Corollary. If $m^* A = 0$, then $m^*(A \cup B) = m^*B$.

Proof. Using the above proposition

$m^* (A \cup B) \leq m^*A + m * B$

$= 0 + m^* B$ (i)

Also $B \subset A \cup B$

Therefore $m^* B \leq m^* (A \cup B)$ (ii)

From (i) and (ii) it follows that

$$m^* B = m^* (A \cup B)$$

Note:- Because of the property $m^* (\cup A_n) \leq \Sigma m^* A_n$, the function m* is said to be countably Sub-additive. It would be much better if m* were also countably additive, that is,

if $m^* (\cup A_n) = \sum m^* A_n$.

for every countable collection $[A_n]$ of disjoint sets of real numbers. If we insist on countable additivity, we have to restrict the domain of the function m* to some subset m of the set 2^R of all subsets of R. The members of m are called the measurable subsets of R. That is, to do so we suitably reduce the family of sets on which m* is defined. This is done by using the following definition due to Carathedory.

1.18 Definition. A set E of real numbers is said to be m* measurable, if for every set A $\in$ R, we have

$$m^* A = m^* (A \cap E) + m^* (A \cap E^c)$$

Since $A = (A \cap E) \cup (A \cap E^c)$,

It follows from the definition that

$$m^* A = m^* [(A \cap E) \cup (A \cap E^c)] \leq m^* (A \cap E) + m^* (A \cap E^c)$$

Hence, the above definition reduces to:

A set E $\in$ R is measurable if and only if for every set A $\in$ R, we have

$m^* A \geq m^* (A \cap E) + m^* (A \cap E^c)$.

For example ϕ is measurable.

1.19 Theorem. Prove that ϕ is measurable set.

Proof: Let A be set of reals, then $m^* A = m^* (A \cap E) + m^* (A \cap E^c)$

Put $E = \phi$

$m^* (A \cap \phi) + m^* (A \cap \phi^c) = m^* (\phi) + m^* (A \cap R)$

$= 0 + m^* A$

$= m^* A$

This implies ϕ is measurable.

1.20 Theorem. Prove that R is measurable set.

Proof: Let A be set of reals, then

$m^* A = m^* (A \cap E) + m^* (A \cap E^c)$

Put E = R

$m^* (A \cap R) + m^* (A \cap R^c) = m^* (A) + m^* (A \cap \phi)$

$= m^* (A) + m^* (\phi)$

$= m^* A + 0$

$= m^* A$

This implies R is measurable.

1.21 Theorem. If $m^* E = 0$, then E is measurable.

Proof. Let A be any set. Then $A \cap E \subset E$ and so

$$m^* (A \cap E) \le m^* E = 0 \qquad \text{(i)}$$

Also $A \supset A \cap E^c$, and so

$$m^* A \ge m^* (A \cap E^c) = m^* (A \cap E^c) + m^* (A \cap E)$$

as $m^* (A \cap E) = 0$ by (i)

Hence E is measurable.

1.22 Theorem. Every subset of E is measurable if $m^* E = 0$.

Proof: Let F be any subset of E, where $m^* E = 0$.

then since $F \subseteq E$

this implies $m^* F \le m^* E$

this implies $m^* F \le 0$

Also $m^* F \ge 0$

therefore $m^* F = 0$.

this implies F is measurable.

1.23 Theorem. Every singleton set is measurable.

Proof: Since outer measure of singleton set is zero and set of measure zero is measurable. Therefore, singleton set is measurable.

1.24 Theorem. Every countable set is measurable.

Proof: Since outer measure of countable set is zero and set of measure zero is measurable. Therefore countable set is measurable.

1.25 Theorem. If a set E is measurable, then so is its complement E^c.

Proof. The definition is symmetrical with respect to E^c, and so if E is measurable, its complement E^c is also measurable.

1.26 Theorem. Union of two measurable sets is measurable.

Proof. Let E_1 and E_2 be two measurable sets and let A be any set. Since E_2 is measurable, we have

$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$ (i)

and since $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup [A \cap E_2 \cap E_1^c]$ (ii)

Therefore by (ii) we have

$m^*[A \cap (E_1 \cup E_2)] \le m^*(A \cap E_1) + m^*[A \cap E_2 \cap E_1^c]$ (iii)

Thus

$m^*[A \cap (E_1 \cup E_2)] + m^*(A \cap E_1^c \cap E_2^c)$

$\le m^*(A \cap E_1) + m^*(A \cap E_2 \cup E_1^c) + m^*(A \cap E_1^c \cap E_2^c)$

$= m^*(A \cap E_1) + m^*(A \cap E_1^c)\ (by\ (i))$

$\le$ m* A (since E_1 is measurable)

i.e. $m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \le m^* A$

Hence $E_1 \cup E_2$ is measurable.

If E_1 and E_2 are measurable, then $E_1 \cap E_2$ is also measurable.

In fact we note that E_1, E_2 are measurable $\Rightarrow E_1^c, E_2^c$ are measurable $\Rightarrow E_1^c \cup E^c$ is measurable $\Rightarrow (E_1^c \cup E_2^c)^c = E_1 \cap E_2$ is measurable.

Similarly, it can be shown that if E_1 and E_2 are measurable, then $E_1^c \cap E_2^c$ is also measurable.

1.27 Lemma. Difference of two measurable sets is also measurable.

Proof: Let E_1 and E_2 be two measurable sets. Then E_2^c is measurable and hence $E_1 \cap E_2^c = E_1 - E_2$ is measurable, being the intersection of two measurable sets.

1.28 Definition. Algebra or Boolean Algebra: - A collection **A** of subsets of a set X is called an algebra of sets or a Boolean Algebra if

(i) A, B ∈ A => A ∪ B ∈ A
(ii) A ∈ A => A^c ∈ A
(iii) For any two members A and B of A, the intersection A ∩ B is in A.

Because of De Morgan's formulae (i) and (ii) are equivalent to (ii) and(iii).

It follows from the above definition that the collection M of all measurable sets is an algebra. The proof is an immediate consequence of Theorems 1.25 and 1.26.

1.29 Definition. By a Boolean σ - algebra or simply a σ - algebra or Borel field of a collection of sets, we mean a Boolean Algebra A of the collection of the sets such that union of any countable collection of members of this collection is a member of A.

From De Morgan's formula an algebra of sets is a σ - algebra or Borel field if and only if the intersection of any countable collection of members of A is a member of A.

1.30 Lemma. Let A be any set, and $E_1, E_2, \dots, E_n$ a finite sequence of disjoint measurable sets. Then

$$m^* (A \cap [U_{i=1}^n E_i]) = \Sigma_{i=1}^n m^*(A \cap E_i)$$

Proof. We shall prove this lemma by induction on n. The lemma is trivial for

n = 1. Let n > 1 and suppose that the lemma holds for n – 1 measurable sets E_i.

Since E_n is measurable, we have

$m^* (X) = m^* (X \cap E_n) + m^* (X \cap E_n{}^c)$ for every set X ∈ R.

In particular we may take

$X = A \cap [U_{i=1}^n E_i]$.

Since $E_1, E_2, \dots, E_n$ are disjoint, we have

$$X \cap E_n = A \cap [U_{i=1}^n E_i] \cap E_n = A \cap E_n$$

$$X \cap E_n^c = A \cap [U_{i=1}^n E_i] \cap E_n^c = A \cap [U_{i=1}^{n-1} E_i]$$

Hence, we obtain $m^* X = m^*(A \cap E_n) + m^*(A \cap [U_{i=1}^{n-1} E_i])$ (i)

But since the lemma holds for $n - 1$ we have

$$m^*(A \cap [U_{i=1}^{n-1} E_i]) = \sum_{i=1}^{n-1} m^*(A \cap E_i)$$

Therefore (i) reduces to

$m^* X = m * (A \cap E_n) + \Sigma_{i=1}^{n-1} m^*(A \cap E_i)$

$= \sum_{i=1}^n m^*(A \cap E_i)$.

Hence the lemma.

1.31 Lemma. Let A be an algebra of subsets and $\{E_i \mid i \in N\}$ a sequence of sets in A. Then there exists a sequence $[D_i \mid i \in N]$ of disjoint members of A such that

$$D_i \subset E_i \ (i \in N)$$

$$U_{i \in N} D_i = U_{i \in N} E_i$$

Proof. For every i $\in$ N, let

$$D_n = E_n - (E_1 \cup E_2 \cup \dots \dots \cup E_{n-1})$$
$$= (E_n \cap (E_1 \cup E_2 \cup \dots \dots \cup E_{n-1}))^c$$
$$= E_n \cap {E_1}^c \cap {E_2}^c \cap \dots \dots \cap {E_{n-1}}^c$$

Since the complements and intersections of sets in A are in A, we have each $D_n \in A$. By construction, we obviously have $D_i \subset E_i$ $(i \in N)$

Let D_n and D_m be two such sets, and suppose m < n. Then $D_m \subset E_m$, and so

$D_m \cap D_n \subset E_m \cap D_n$

$= E_m \cap E_n \cap {E_1}^c \cap \dots . . E_m^c \cap \dots . \cap E_{n-1}^c \; \big(using\ (i)\big)$

$= (E_m \cap E_m^c) \cap \dots . = \phi \cap \dots \dots \dots = \phi$

The relation (i) implies $U_{i \in N} D_i \subset U_{i \in N} E_i$

It remains to prove that

$U_{i \in N} D_i \supset U_{i \in N} E_i$

For this purpose let x be any member of $U_{i \in N} E_i$. Let n denotes the least natural number satisfying x $\in$ E_n. Then we have

$x \in E_n - (E_1 \cup E_2 \cup \dots \dots \cup E_{n-1}) = D_n \subset U_{i \in N} D_n.$

This completes the proof.

1.32 Theorem. The collection M of measurable sets is a σ - algebra.

Proof. We have proved already that M is an algebra of sets and so we have only to prove that M is closed with respect to countable union. By the lemma proved above each set E of such countable union must be the union of a sequence $\{D_n\}$ of pairwise disjoint measurable sets. Let A be any set, and let

$E_n = U_{i \in I} D_i \subset E$. Then E_n is measurable and $E_n^c \supset E^c$. Hence

$m^* A = m^*(A \cap E_n) + m^*(A \cap E_n^c) \geq m^*(A \cap E_n) + m^*(A \cap E_n^c).$

But, by lemma 1.30,

$$m^*(A \cap E_n) = m^*[A \cap (U_{i \in 1} D_i)] = \Sigma_{i=1}^n m^*(A \cap D_i)$$

Therefore,

$m^* A \geq \Sigma_{i=1}^n m^*(A \cap D_i) + m^*(A \cap E^c)$

Since the left hand side of the inequality is independent of n, we have

$$m^* A \geq \Sigma_{i=1}^\infty m^*(A \cap D_i) + m^*(A \cap E^c)$$

$\geq m^*(U_{i \in I}^\infty [A \cap D_i]) + m^*(A \cap E^c)$ (by countably subadditivity of m*)

$$= m^*(A \cap U_{i \in I}^\infty D_i) + m^*(A \cap E^c)$$

$$= m^*(A \cap E) + m^*(A \cap E_n^c).$$

which implies that E is measurable. Hence the theorem.

1.33 Lemma. The interval (a, ∞) is measurable

Proof. Let A be any set and

$$A_1 = A \cap (a, \infty)$$

$$A_2 = A \cap (a, \infty)^c = A \cap (-\infty, a].$$

Then we must show that

$$m^* A_1 + m^* A_2 \leq m^* A.$$

$If\ m^* A = \infty$, then there is nothing to prove. If m* A < ∞ , then given ∈ > 0 there is a countable collection $\{I_n\}$ of open intervals which cover A and for which

$$\Sigma l(I_n) \leq m^* A + \in$$

Let $I_n' = I_n \cap (a, \infty)\ and\ I_n'' = I_n \cap (-\infty, a)$. Then I_n' and I_n'' are intervals (or empty) and $l(I_n) = l\,(I_n') + l(I_n'') = m^*(\,I_n') + m^*(\,I_n'')$

Since $A_1 \subset UI_n'$, we have

$m^* A_1 \leq m^*(UI_n') \leq \Sigma\, m^* I_n'$, (iii)

and since, $A_2 \subset U\, I_n''$, we have

$m^* A_2 \leq m^*(U\, I_n'') \leq \Sigma\, m^* I_n''$, (iv)

Adding (iii) and (iv) we have

$m^* A_1 + m^* A_2 \leq \Sigma\, m^* I_n' + \leq \Sigma\, m^* I_n''$

$= \Sigma\,(m^* I_n' + \leq m^* I_n'')$

$= \Sigma\, l\,(I_n) \qquad [by\ (ii)]$

$\leq m^* A + \in \quad [by\ (i)]$

But ∈ was arbitrary positive number and so we must have $m^* A_1 + m^* A_2 \leq m^* A$.

1.34 Definition. The collection ß of Borel sets is the smallest σ - algebra which contains all of the open sets.

1.35 Theorem. Every Borel set is measurable. In particular each open set and each closed set is measurable.

Proof. We have already proved that (a, ∞) is measurable. So we have

$(a, \infty)^c = (-\infty, a]$ measurable.

Since $(-\infty, b) = U_{n=1}^{\infty} \left((-\infty, b - \frac{1}{n}]\right)$ and we know that countable union of measurable sets is measurable, therefore (- ∞, b) is also measurable. Hence each open interval,

$(a, b) = (-\infty, b) \cap (a, \infty)$ is measurable, being the intersection of two measurable sets. But each open set is the union of countable number of open intervals and so must be measurable (The measurability of closed set follows because complement of each measurable set is measurable).

Let M denote the collection of measurable sets and C the collection of open sets. Then

C ⊂ M. Hence ß is also a subset of M since it is the smallest σ - algebra containing C . So each element of ß is measurable. Hence each Borel set is measurable.

1.36 Definition. If E is a measurable set, then the outer measure of E is called the Lebesgue Measure of E, is denoted by m. Thus, m is the set function obtained by restricting the set function m* to the family M of measurable sets. Two important properties of Lebesgue measure are summarized by the following theorem.

1.37 Theorem. Let $\{E_n\}$ be a sequence of measurable sets. Then

$m(\cup E_i) \leq \Sigma\, m\, E_i$

If the sets E_n are pairwise disjoint, then

$$m(\cup E_i) = \Sigma\, mE_i\,.$$

Proof. The inequality is simply a restatement of the sub-additivity of m*. If $\{E_i\}$is a finite sequence of disjoint measurable sets. So we apply lemma 1.30 replacing A by R. That is , we have

$$m^*(R \cap [U_i^n E_i]) = \Sigma_{i=1}^n\, m^*\,(R \cap E_i)$$

$$m^*(U_i^n E_i) = \sum_i^n m^* E_i$$

and so m is finitely additive..

Let $\{E_i\}$ be an infinite sequence of pairwise disjoint sequence of measurable sets. Then

And so $\quad U_{i=1}^\infty\, E_i \supset U_{I=1}^n E_n$

$m(U_{i=1}^\infty\, E_i) \geq m(U_{(i=1)}^\infty\, E_i) = \Sigma_{i=1}^\infty\, m\, E_i$

Since the left-hand side of this inequality is independent of n, we have

$m(U_{i=1}^\infty\, E_i) \geq \Sigma_{i=1}^\infty\, m\, E_i$

The reverse inequality follows from countable sub-additivity and we have

$m(U_{i=1}^\infty\, E_i) = \Sigma_{i=1}^\infty\, m\, E_i$

Hence the theorem is proved.

1.38 Theorem. Let $\{E_n\}$ be an infinite sequence of measurable sets such that $E_{n+1} \subset E_n$ for each n. Let $mE_1 < \infty$. Then

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} mE_n$$

Proof. Let $E = \bigcap_{i=1}^{\infty} E_i$ and let $F_i = E_i - E_{i-1}$. Then since $\{E_n\}$ is a decreasing sequence. We have $\bigcap F_i = \phi$.

Also we know that if A and B are measurable sets then their difference $A - B = A \cap B^c$ is also measurable. Therefore each F_i is measurable. Thus $\{F_i\}$ is a sequence of measurable pairwise disjoint sets.

$$\text{Now} \qquad \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (E_i - E_{i+1})$$

$$= \bigcup_{i=1}^{\infty} (E_i \cap E_{i+1}^c)$$

$$= E_1 \cap (\cup\ E_i^c)$$

$$= E_1 \cap \left(\bigcap_{i=1}^{\infty} E_i\right)^c$$

$$= E_1 \cap E^c$$

$$= E_1 - E$$

Hence

$$m\left(\bigcup_{i=1}^{\infty} F_i\right) = m(E_1 - E)$$

$$\Rightarrow \sum_{i=1}^{\infty} mF_i = m(E_1 - E)$$

$$\Rightarrow \textstyle\sum_{i=1}^{\infty} m(E_i - E_{i+1}) = m(E_1 - E) \qquad \ldots \quad \text{(i)}$$

Since $E_1 = (E_1 - E) \cup E$, therefore

$$mE_1 = m(E_1 - E) + m(E)$$

$$\Rightarrow mE_1 - mE = m(E_1 - E) \quad (\text{since } mE \le mE_1 < \infty\ ) \ \ldots \text{(ii)}$$

Again

$$E_i = (E_i - E_{i+1}) \cup E_{i+1}$$

$$\Rightarrow mE_i = m(E_i - E_{i+1}) + mE_{i+1}$$

$$\Rightarrow mE_i - mE_{i+1} = m(E_i - E_{i+1}) \ (\text{ since } E_{i+1} \subset E_i\ ) \ldots \text{(iii)}$$

Therefore (i) reduces to

$$mE_1 - mE = \textstyle\sum_{i=1}^{\infty} (mE_i - mE_{i+1}) \ \ (\text{ using (ii)and (iii)})$$

$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} (mE_i - mE_{i+1})$$

$$= \lim_{n\to\infty} [mE_1 - mE_2 + mE_2 - mE_3 \ldots - mE_{n+1}]$$

$$= \lim_{n\to\infty} [mE_1 - mE_{n+1}]$$

$$= mE_1 - \lim_{n\to\infty} E_{n+1}$$

$$\Rightarrow mE = \lim_{n\to\infty} mE_n$$

$$\Rightarrow m(\cap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} mE_n$$

1.39 Remark. Show that the condition $m(E_1) < \infty$ is necessary in the above theorems.

Solution. Let $E_n = [n, \infty)$

Then, $E_1 = [1, \infty)$

$\Longrightarrow m(E_1) = m[1, \infty) = \infty$

We show that the proposition of decreasing sequence does not hold in this case i.e. we want to show that

$m(\cap_{n=1}^{\infty} E_n) \neq \lim_{n\to\infty} (E_n)$

Clearly, $E_{n+1} \subset E_n$ for all n

Now, $E_n = [n, \infty) \supset [n, 2n]$

$\Longrightarrow \quad m(E_n) \geq m[n, 2n] = n$

$\Longrightarrow \quad m(E_n) \geq n$

$\Longrightarrow \lim_{n\to\infty} m(E_n) = \infty \qquad \ldots \qquad (1)$

Now, we claim that $m(\cap_{n=1}^{\infty} E_n) = m(\cap_{n=1}^{\infty} [n, \infty)) = 0$

For if, $\cap_{n=1}^{\infty} E_n \neq \phi \Longrightarrow$ there exists $x \in \cap_{n=1}^{\infty} E_n$

$\Longrightarrow \quad x \in [n, \infty)$ for all $n \in N$

Let $x \in R$, so by Archmedian property, we can find a positive integer n_0 such that

$$n_0 \leq x < n_0 + 1$$

$\Longrightarrow \quad x \notin [n_0 + 1, \infty)$, a contradiction

$\therefore \cap_{n=1}^{\infty} E_n = \phi$

$\Longrightarrow \quad m(\cap_{n=1}^{\infty} E_n) = 0 \qquad \ldots \qquad (2)$

From (1) and (2), we have

$$m(\cap_{n=1}^{\infty} E_n) \neq \lim_{n\to\infty} (E_n)$$

S0, theorem does not hold in this case.

1.40 Theorem. Let $\{E_n\}$ be an increasing sequence of measurable sets. i.e. a sequence with $E_n \subset E_{n+1}$ for each n. Let mE_1 be finite, then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n \,.$$

Proof. The sets E_1, $E_2 - E_1$, $E_3 - E_2$, …, $E_n - E_{n+1}$ are measurable and are pairwise disjoint . Hence

$$E_1 \cup (E_2 - E_1) \cup \dots \cup (E_n - E_{n-1}) \cup \dots$$

is measurable and

$$m[E_1 \cup (E_2 - E_1) \cup \dots \cup (E_n - E_{n-1}) \cup \dots]$$

$$= mE_1 + \sum_{i=2}^{n} m\,(E_i - E_{i-1})$$

$$= mE_1 + \lim_{n\to\infty} \sum_{i=2}^{n} m(E_i - E_{i-1})$$

But

$E_1 \cup (E_2 - E_1) \cup \dots \cup (E_n - E_{n-1}) \cup \dots$ is precisely $\bigcup_{i=1}^{\infty} E_n$

Moreover,

$$\sum_{i=2}^{n} m\,(E_i - E_{i-1}) = \sum_{i=2}^{n} (mE_i - mE_{i-1})$$

$$= (mE_2 - mE_1) + (mE_3 - mE_2) + \cdots + (mE_n - mE_{n-1})$$

$$= mE_n - mE_1$$

Thus we have

$$m\left[\bigcup_{i=1}^{\infty} E_i\right] = mE_1 + \lim_{n\to\infty}[mE_n - mE_1]$$

$$= \lim_{n\to\infty} mE_n$$

1.41 Definition : The symmetric difference of the sets A and B is the union of the sets A-B and B-A . It is denoted by $\Delta\, B$.

1.42 Theorem. If $m(E_1 \,\Delta\, E_2) = 0$ and E_1 is measurable, then E_2 is measurable. Moreover $mE_2 = mE_1$.

Proof . We have

$E_2 = [E_1 \cup (E_2 - E_1)] - (E_1 - E_2)$ …(i)

By hypothesis, both $E_2 - E_1$ and $E_1 - E_2$ are measurable and have measure zero. Since E_1 and E_2-E_1 are disjoint, $E_1 \cup (E_2 - E_1)$ is measurable and

$m[E_1 \cup (E_2 - E_1)] = mE_1 + 0 = mE_1$. But, since

$E_1 - E_2 \subset [E_1 \cup (E_2 - E_1)]$,

it follows from (i) that E_2 is measurable and

$$\begin{aligned} mE_2 &= m[\, E_1 \cup (E_1 - E_2)] - m(E_1 - E_2) \\ &= mE_1 - 0 = mE_1 \,. \end{aligned}$$

This completes the proof.

1.43 Definition. Let x and y be real numbers in [0,1]. The **sum modulo 1 of x and y** , denoted by $x \overset{0}{+} y$, is defined by

$$x \overset{0}{+} y = \begin{cases} x + y \; if \; x + y < 1 \\ x + y - 1 \; if \; x + y \geq 1 \end{cases}$$

It can be seen that $\overset{0}{+}$ is a commutative and associative operation which takespair of numbersin [0,1) into numbers in [0,1).

If we assign to each $x \in [0,1)$ the angle $2\pi x$ then addition modulo 1 corresponds to the addition of angles.

If E is a subset of [0,1), we define the translation modulo 1 of E to be the set

$E \overset{0}{+} y$ = [z $|z =$ x $\overset{0}{+}$ y for some x $\in E$].

If we consider addition modulo 1 as addition of angles, translation module 1 by y corresponds to rotation through an angle of $2\pi y$.

We shall now show that Lebesgue measure is invariant under translation modulo 1.

1.44 Definition. Let x and y be real numbers in [0,1). The **sum modulo 1 of x and y**, denoted by $\overset{0}{+} y$, is defined by

$$x \overset{0}{+} y = \begin{cases} x + y \; if \; x + y < 1 \\ x + y - 1 \; if \; x + y \geq 1 \end{cases}$$

Clearly $x \overset{0}{+} y \in [0,1)$

It can be seen that $\overset{0}{+}$ is a commutative and associative operation which takes pair of numbers in [0,1) into numbers in [0,1).

1.45 Definition. If E is a subset of [0,1), we define the translation modulo 1 of E to be the set

$E \overset{0}{+} y$ = [z |$z =$ x $\overset{0}{+}$ y for some x ∈ E].

We shall now show that Lebesgue measure is invariant under translation modulo 1.

1.46 Lemma. Let $E \subset [0,1)$ be a measurable set. Then for each $y \in [0,1)$ the set $E \overset{0}{+} y$ is measurable and m $(E \overset{0}{+} y) = mE.$

Proof. Let E_1= E$\cap [0, 1 - y)$ and E_2 = E $\cap [1 - y, 1)$. Then E_1 and E_2 are disjoint measurable sets whose union is E, and so, mE = mE_1 + mE_2.

we observe that

$E_1 \overset{0}{+} y$ ={ $x \overset{0}{+} y$: x ∈ E_1}

$$= \begin{cases} x + y \; if \; x + y < 1 \\ x + y - 1 \; if \; x + y \geq 1. \end{cases} \qquad x \in E_1$$

But for x ∈ E_1, we have x + y < 1 and so

$E_1 \overset{0}{+} y = \{ x + y, x \in E_1\} = E_1 + y.$

and hence $E_1 \overset{0}{+} y$ is measurable. Thus

m $(E_1 \overset{0}{+} y) = m(E_1 + y) = m(E_1),$

since m is translation invariant. Also $E_2 \overset{0}{+} y = E_2 + (y - 1)$ and so $E_2 \overset{0}{+} y$ is measurable and m$(E_2 \overset{0}{+} y) = mE_2$. But

$$E \overset{0}{+} y = (E_1 \overset{0}{+} y) \cup (E_2 \overset{0}{+} y)$$

And the sets $(E_1 \overset{0}{+} y)$ and $(E_2 \overset{0}{+} y)$ are disjoint measurable sets. Hence $E \overset{0}{+} y$ is measurable and

$$m\,(E \overset{0}{+} y) = m[(E_1 \overset{0}{+} y) \cup (E_2 \overset{0}{+} y)]$$

$$= m(E_1 \overset{0}{+} y) + m(E_2 \overset{0}{+} y)$$

$$= m(E_1) + m(E_2)$$

$$= m(E).$$

This completes the proof of the lemma.

1.47 Theorem: Prove that there exists a non-measurable set in interval [0,1).

Proof: First we define an equivalence relation in the set I= [0,1), By saying that x and y are equivalent i.e., $x \sim y$ if and only if x-y is a rational number.

If x-y is a rational number, we say that x and y are equivalent and write x-y. It is clear that $x \sim x\,; x \sim y \Longrightarrow y \sim x \; and \; x \sim y\,, y \sim z \; \Longrightarrow x \sim z$. Thus ‘ $\sim$’ is an equivalence relation in I.

Hence the relation ~ partitions the set I = [0,1) into mutually disjoint equivalence classes, that is, classes such that any two elements of one class differ by a rational number, while any two elements of different classes differ by an irrational number.

Construct a set P by choosing exactly one element from each equivalence classes. Now we claim that P is a non-measurable set.

Let $< r_i > \substack{\infty \\ i=0}$ be a sequence of the rational numbers in [0,1) with $r_0 = 0$ and define $P_i = P \overset{0}{+} r_i$.

(translation modulo 1 of P)

Then $P_0 = P$.

We further prove that (i) $P_i \cap P_j = \emptyset, i \neq j$.

(ii) $\bigcup_n P_n = [0, 1)$

Proof: (i) Let $P_i \cap P_j \neq \emptyset, i \neq j$.

Let $x \in P_i \cap P_j$. => $x \in P_i and \; x \in P_j$

Then$\exists \; p_i, p_j \in P \; such \; that \; x = p_i \overset{0}{+} r_i$

$$x = p_j \overset{0}{+} r_j$$

- ⇨ $p_i \overset{0}{+} r_i = p_j \overset{0}{+} r_j$
- ⇨ p_i-$p_j = r_j - r_i$ *is a rational number.*
- ⇨ $p_i \sim p_j$ *is a rational number.*

  i.e., $p_i \sim p_j$

  => p_i *and* p_j *are in same equivalence class.*

But P has only one element from each equivalence class, therefore we must have $p_i = p_j$ $i.e., i = j$

But $\neq j$. Hence a contradiction.

Hence $P_i \cap P_j \neq \emptyset, i \neq j$.

that is, < P_i >is a pair wise disjoint sequence of sets.

(ii) Clearly each $P_i \subset [0, 1)$

$\bigcup_i P_i \subset [0, 1)$. Let x be any element of [0, 1) = I.

But I is partitioned into equivalent classes therefore x lies in one of the equivalence classes.

- ⇨ x is equivalent to an element say y of P.
- ⇨ x-y is a rational number say r_i.
- ⇨ x-y = r_i
- ⇨ x = y + r_i

  $= y \overset{0}{+} r_i$.

  $x \in P \overset{0}{+} r_i$

- ⇨ x$\in P_i$
- ⇨ x is in some P_i.

$$Therefore\ [0, 1) \subseteq \bigcup_i P_i$$

$$therefore, [0, 1) = \bigcup_i P_i.$$

Now we prove P is non-measurable.

Assume that P is measurable, then clearly each P_i is measurable.

And $m(P_i) = m\left(P \overset{0}{+} r_i\right)$

$= m(P)$ for each i.

Therefore, $m(\bigcup_i P_i) = \sum_i m(P_i) = \sum_{i=0}^{\infty}(P)$

$$= \begin{cases} 0 & if\ m(P) = 0 \\ \infty & if\ m(P) > 0 \end{cases}$$

But

$$m\left(\bigcup_i P_i\right) = m[(0,1)] = l(0,1) = 1, contadiction.$$
$$Therefore\ P\ is\ non-measurable\ set.$$

1.48 Example. The cantor set is uncountable with outer measure zero.

Solution. We already know that cantor set is uncountable. Let C_n denote the union of the closed intervals left at the nth stage of the construction. We note that C_n consists of 2^n closed intervals, each length 3^{-n}. Therefore

$$m^*C_n \leq 2^n.3^{-n} \qquad (\therefore\ m^*C_n = m^*(\cup F_n) = \sum m^*F_n)$$

But any point of the cantor set C must be in one of the intervals comprising the union C_n, for each n $\in N$, and as such $C \subset C_n$ for all n $\in N$. Hence

$$m^*C \leq m^*C_n \leq \left(\frac{2}{3}\right)^n$$

This being true for each n $\in N$, letting $n \to \infty$ gives $m^*C = 0$.

1.49 Example. If E_1 and E_2 are any measurable sets, show that

$$M(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Proof. Let A be any set. Since E_1 is measurable,

$$m^*A = m^*(A \cap E_1) + m^*(A \cap {E_1}^c).$$

We set $A = E_1 \cup E_2$ and we have

$$m^*(E_1 \cup E_2) = m^*[(E_1 \cup E_2) \cap E_1] + m^*[(E_1 \cup E_2) \cap {E_1}^c]$$

Adding $m(E_1 \cup E_2)$ to both sides we have

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + m^*[(E_1 \cup E_2) \cap {E_1}^c] + m(E_1 \cap E_2) \ \ldots (1)$$

But

$$E_2 = [(E_1 \cup E_2) \cap {E_1}^c] \cup (E_1 \cup E_2).$$

Therefore

$$m\{[(E_1 \cup E_2) \cap {E_1}^c] \cup (E_1 \cup E_2)\} = mE_2$$

Hence (1) reduces to

$$M(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

1.50 Theorem. Let E be any set. Then given $\in > 0$, there is an open set $O \supset E$ such that $m^*O < m^*E + \in$.

Proof. There exists a countable collection [I_n] of open intervals such that

$E \subset \bigcup_n I_n$ and

$$\sum_{n=1}^{\infty} l(I_n) < m^* E + \in .$$

$$put\ O = \bigcup_{n=1}^{\infty} I_n.$$

Then O is an open set and

$$m^*O = m^*\left(\bigcup_{n=1}^{\infty} I_n\right)$$

$$\leq \sum_{n=1}^{\infty} m^* I_n$$

$$= \sum_{n=1}^{\infty} l(I_n) < m^*E + \in .$$

1.51 Theorem. Let E be a measurable set. Given $\in > 0$, there is an open set

$O \supset E$ such that $m^*(O \backslash E) < \in$.

Proof. Suppose first that m E < ∞. Then by the above theorem there is an open set $O \supset E$ such that

$$m^*O < m^*E + \in$$

Since the sets O and E are measurable, we have

$$m^*(O \backslash E) = m^*O - m^*E < \in.$$

Consider now the case when m E = ∞ . Write the set **R** of real number as a union of disjoint finite intervals; that is,

$$\mathbf{R} = \bigcup_{n=1}^{\infty} \mathbf{I_n}.$$

Then, if $E_n = E \cap I_n$, $m(E_n) < \infty$. We can, thus, find open sets $O_n \supset E_n$ such that

$$m^*(O_n - E_n) < \frac{\in}{2^n}.$$

Define $O = \bigcup_{n=1}^{\infty} O_n$. Clearly O is an open set such that $O \supset E$ and satisfies

$$O - E = \bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} (O_n - E_n)$$

$$m^*(O - E) \leq \sum_{n=1}^{\infty} m^*\left(\frac{O_n}{E_n}\right) < \in .$$

1.52 F_σ and G_δ Sets:

A set which is countable(finite or infinite) union of closed sets is called an F_σ sets. Note: The class of all $\boldsymbol{F_\sigma}$ sets is denoted by F_σ. This **F** stands for ferme(closed) and σ for summe(sum).

Example:

1. A closed set.
2. A countable set
3. A countable union of F_σ set.
4. An open interval (a, b) since

   $(a, b) = U_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$and hence an open set.

***G_δ*- set:**

A set which is countable intersection of open sets is a G_δ set.

Note: The class of all G_δ sets is denoted by This G stands for region and $\delta$for intersection. The complement of $\boldsymbol{F_\sigma}$ set is a $\boldsymbol{G_\delta}$ set and conversely.

Example:

1. An open set in particular an open interval.
2. A closed set
3. A countable intersection of G_δ set.
4. A closed interval [a, b] since

   $[a, b] = \cap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.

1.53 Theorem. Let E be any set then

(a) Given ε > 0,∃an open set $O \supset E$such that $m^*(O) < m^*(E) + \varepsilon$

$(b) \exists a\ G_\delta\ set\ G \supset E$such that m*(E) = m*(G).

Proof: (a) By definition, m* $(E) = inf\ \sum_n l\ (I_n)$, where $E \subseteq \bigcup_n I_n$

if m*(E) = ∞, then clearly result is true. If m*(E) < 0, there is a countable collection {In} of open intervals such that

$E \subseteq U_n\ I_n$and $m^*(E) + \varepsilon > \sum_n l\ (I_n)$ (1)

Let $O = E \subseteq U_n\ I_n$, then O is an open set and $O \supset E$

Also $m^*(O) = m^*(E \subseteq U_n\ I_n)$

$$\leq \sum_n m^*(I_n)$$

m*(O) < m*(E) + ε [from(1)]

(b) Take ε $= \frac{1}{n}$ $\forall n \in N$ Then by above part, for each $n \in N$,∃an open set$O_n \supset E$such that

$$m^*(O_n) < m^*(E) + \frac{1}{n}$$

Now define $G = \cup_{n=1}^{\infty} O_n$, then G is a G_δ set.

Also, since each $O_n \supset$ E

therefore $\cup_{n=1}^{\infty} O_n \supset E$ this implies $G \supset E$

$\geq$ m*(E) ≤ m*(G) (2)

Also $G = \cup_{n=1}^{\infty} O_n \subseteq O_n \; \forall n$

$m^*(G) \leq m^*(O_n)$ for each n

$< m^*(E) + \frac{1}{n}$, for each n

in limiting case, we have

m*(G) ≤ m*(E) (3)

Then from (2) and (3), we have

m*(G) = m*(E).

1.54 Theorem. Let E be any set,then the following five statements are equivalent.

(i) E is measurable.

(ii) For given **ε** > 0 , ∃an open set $O \supset E$ such that **m*(O – E) < ε**

(iii) There exist a set G in G_δ with $E \subset G$, m*(G – E) = 0

(iv) For given ε > 0 , ∃ an closed set $F \subset E$ such that **m*(E – F) < ε**

(v) There exist a set F in F_σ with $F \subset E$, m*(E – F) = 0

Proof. Ist we prove (i) implies (ii)

Let E be a measurable set.

Now two cases arrive

Case (i) m*(E) < ∞.

By definition, for given **ε**> 0,there is a countable collection $\{I_n\}$ of open intervals such that

$E \subseteq \bigcup_n I_n$ and m*(E) + ε > $\sum_n l\,(I_n)$.......(1)

Let O = $E \subseteq \bigcup_n I_n$, then O is an open set and $O \supset E$

Also m*(O) =m*($E \subseteq \bigcup_n I_n$)

m*(O) $\leq \sum_n m*(I_n)$

m*(O) < m*(E) + ε [from(1)]

m*(O)- m*(E) < ε

O = $(O - E) \cup E$

m*(O) = m*($(O - E) \cup E$) = m*(O-E)+m*(E)

m*(O-E) = m*(O) - m*(E)

$\Longrightarrow$ m*(O-E) < ε

Case (ii) If m*(E) = ∞

We know that set of real number can be written as countable union of disjoint open intervals

$$R = \bigcup_{n=1}^{\infty} I_n$$

Then $E = E \cap R$

$$= E \bigcap \bigcup_{n=1}^{\infty} I_n$$

$$= \bigcup_{n=1}^{\infty} (E \cap I_n)$$

$$\Longrightarrow E = \bigcup_{n=1}^{\infty} E_n \text{ , } where\ E_n = E \cap I_n$$

clearly each E_n is measurable and m(E_n) is finite.

Because $E_n = E \cap I_n \subseteq I_n$

m*(E_n) ≤ l(I_n) <∞

Then by case (i), for each $n \in N$, ∃ an open set $O_n \supset E_n$ such that

m*(O_n - E_n) <$\frac{\varepsilon}{2^n}$

Let us define $O = \bigcup_{n=1}^{\infty} O_n$

Then O is an open set containing E

Now (O-E) = $\bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} (O_n - E_n)$

m*(O-E) ≤ m*($\bigcup_{n=1}^{\infty} (O_n - E_n)$)

$\leq \sum_{n=1}^{\infty} m^* (O_n - E_n)$

$\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$

$= \varepsilon$

$\Longrightarrow$ m*(O-E) < ε

Now (ii) $\Longrightarrow$ (iii)

Let (ii) holds, then for each $n \in N$, $\exists$an open set $O_n \supset E$such that

m*(O_n-E) <$\frac{1}{n}$

Let us define = $\bigcap_{n=1}^{\infty} O_n$, then G is a $G_\delta$set.

Also since each $O_n \supset E$

therefore $\bigcap_{n=1}^{\infty} O_n \supset E$ this implies $G \supset E$

G-E = $\bigcap_{n=1}^{\infty} O_n - E \subseteq O_n - E$

m*(G-E) ≤ m*(O_n-E) < $\frac{1}{n}$

Since n is arbitrary

m*(G-E) ≤ 0

$\Longrightarrow$ m*(G-E) = 0.

Now (iii) $\Longrightarrow$ (i)

Let (iii) holds, then for given set E, $\exists$ a G_δ set $G \supset E$ such that m*(G-E) = 0

$\Longrightarrow$ G - E is measurable.

Now E = G- (G-E)

Now E is measurable being difference of two measurable sets.

Thus (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

Now to show (i) $\Longrightarrow$ (iv)

Let (i) holds, and ε > 0 be given

then by (ii), for given set E^c, $\exists$ an open set $G \supset \overset{c}{E}$ such that m*(G - E^c) < ε

Since $G \supset \overset{c}{E} \Longrightarrow \overset{c}{G} \subseteq E$

Let F = G^c

then F is a closed set contained in E,

Now E-F = $E \cap \overset{c}{F}$ = E∩G = G∩E = G – E^c

Now m*(E-F) = m*(G-E^c) < ε

m*(E-F) < ε.

To Show (iv) $\Longrightarrow$ (v)

Let (iv) holds, then for each $n \in N$,$\exists$a closed set $F_n \subset E$ such that

m*(E – F$_n$) <$\frac{1}{n}$

Let us define $F = \bigcup_{n=1}^{\infty} F_n$

Then F is a F_σ set.

Also, since each $F_n \subset E \Longrightarrow \bigcup_{n=1}^{\infty} F_n \Longrightarrow F \subseteq E$

Now E – F = E - $\bigcup_{n=1}^{\infty} F_n \subseteq E - F_n$

$\Longrightarrow$ m*(E-F) ≤ **m*(E – F$_n$) <** $\frac{1}{n}$

$\Longrightarrow$ m*(E-F) ≤ $\frac{1}{n}$

Since n is arbitrary.

m*(E-F) ≤ 0

$\Longrightarrow$ m*(E-F) = 0.

Now (v) $\Longrightarrow$ (i)

Let (v) holds, then for general E, ∃ a F_σ set F such that m*(E-F) = 0

$\Longrightarrow$ E-F is measurable.

$$E = (E - F) \cup F$$

$\Longrightarrow$ E is measurable.

This completes the proof.

(b) Take ε = $\frac{1}{n}$ $\forall n \in N$

Then by above part, for each $n \in N$, ∃ an open set $O_n \supset E$ such that

m*(O$_n$) < m*(E) + $\frac{1}{n}$

Now define $G = \bigcap_{n=1}^{\infty} O_n$, then G is a $G_{\delta -}$set.

Also since each $O_n \supset E$

therefore$\bigcap_{n=1}^{\infty} O_n \supset E$

this implies $G \supset E$

$\Longrightarrow$ m*(E) ≤ m*(G) ......(2)

Also G = $\bigcap_{n=1}^{\infty} O_n \subseteq O_n \forall n$

m*(G) ≤ m*(O$_n$) for each n < m*(E) +$\frac{1}{n}$, for each n in limiting case, we have

m*(G) ≤ m*(E)...(3)

Then from (2) and (3), we have

m*(G) = m*(E).

1.55 Theorem. Let E be a set with $m^* E < \infty$. Then E is measurable iff given $\in > 0$, there is a finite union B of open intervals such that $m^*(E \Delta B) < \in$.

Proof. Suppose E is measurable and let $\in > 0$ be given. The (as already shown) there exists an open set $O \supset E$ such that $m^* (O- E) < \frac{\in}{2}$. As m*E is finite, so is m*O. Since the open set O can be written as the union of countable (disjoint) open intervals {Ii}, there exists an $n \in$ **N** such that

$\sum_{i=n+1}^{\infty} l(I_i) < \frac{\in}{2}$ (In fact $m^* O = = \sum_{i=n+1}^{\infty} l(I_i) < \infty \Rightarrow \sum_{i=n+1}^{\infty} l(I_i) < \frac{\in}{2}$ because $m^* O < \infty$)

Set $B = \bigcup_{i=1}^{n} I_i$. . Then $E \Delta B = (E - B) \cup (B - E) \subset (O - B) \cup (O - E)$. Hence

$$m^*(E \Delta B) \le m^* \left(\bigcup_{i=1}^{n} I_i\right) + m^*(O-E) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely, assume that for a given $\in > 0$, there exists a finite union $B = \bigcup_{i=1}^{n} I_i$. if open intervals with $m^* (E \Delta B) < \in$. Then using "Let $\in$ be any set. The given $\in > 0$ there exists an open set $O \supset E$ such that $m^* O < m^* E + \in$ there is an open set $O \supset E$ such that

$$m^* O < m^* E + \in \qquad \text{(i)}$$

If we can show that $m^* (O — E)$ is arbitrary small, then the result will follow from "Let E be

set. Then the following are equivalent (i) E is measurable and (ii) given $\in > 0$ there is an open set $O \supset E$ such that $m^* (O — E) < \in$". Write $S = \bigcup_{i=1}^{n} (I_i \cap O)$. Then $S \subset B$ and so

$S \Delta E = (E - S) \cup (S - E) \subset (E — S) \cup (B — E)$. However,

$E \setminus S = (E \cap O^C) \cup (E \cap B^C) = E — B$, because $E \subset O$. Therefore

$S \Delta E \subset (E — B) \cup (B — E) = E \Delta B$, and as such $m^* (S \Delta E) < \in$. However,

$E \subset S \cup (S \Delta E)$

and so $m^* E < m^* S + m^* (S \Delta E)$

$$< m^*S + \in \qquad \text{(ii)}$$

Also,

$$O — E = (O - S) \cup (S \Delta E)$$

Therefore

$$\begin{aligned} m^* (O \setminus E) &< m^* O - m^* S + \in \\ &< m^* E + \in - m^* S + \in && \text{(using(i))} \\ &< m^* S + \in + \in - m^*S + \in && \text{(using(ii))} \\ &< m^*S + \in + \in - m^* S + \in \\ &= 3 \in . \end{aligned}$$

Hence E is measurable.